*Published:* May 7, 2014
*Last Updated:* August 29, 2017

# KHAYYAM, OMAR XIV. AS MATHEMATICIAN

Three mathematical treatises of Omar Khayyam have come down to us: (1) a commentary on Euclid's *Elements*; (2) an essay on the division of the quadrant of a circle; and (3) a treatise on algebra. A fourth treatise, on the extraction of the *n*th root of numbers, is not extant.

(1) THE COMMENTARY ON EUCLID'S *ELEMENTS*

Khayyam's commentary on the difficulties of certain postulates of Euclid's work (*Resāla fi šarḥ mā aškala men moṣādarāt ketāb Oqlides*) was completed in 470/December 1077. In this treatise, Khayyam intends to amend and rectify what he considers to be the most important difficulties found in the *Elements of Geometry*, or simply the *Elements*, a work in thirteen books attributed to Euclid of Alexandria (fl. ca. 300 BCE). The first part of Khayyam's commentary deals with the theory of parallel lines, the second with the concepts of ratio and proportionality, and the third with the compounding of ratios.

*Theory of parallels*. Euclid (Oqlides) had expounded the theory of parallels in the first book of the *Elements*. In it, he defined parallel lines as "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction" (Heath, I, p. 154). Yet an important part of the theory was based on a statement that Euclid had postulated in the beginning of the same book, that is, the Parallel Postulate:

“That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles” (Heath, I, p. 155). For nearly two thousand years, mathematicians were dissatisfied with this statement, which they always considered as a proposition to be demonstrated rather than a postulate to be admitted.

The first criticisms of the Euclidean theory of parallels have been preserved in the *Commentary on the First Book of Euclid’sElements* by the Neo-Platonist philosopher Proclus Lycius (410-85). Lycius remarks that Posidonius of Rhodes (135-51 BCE) had defined parallel lines as “lines in a single plane which neither converge nor diverge but have all the perpendiculars equal that are drawn to one of them from points on the other” (Proclus, p. 138), that is, as equidistant straight lines. Proclus also mentions an attempt by Ptolemy (fl. ca. 125-61) to prove Proposition I.29 of the *Elements*, the first proposition in which Euclid had made use of the Parallel Postulate, but without resorting to it. Proclus himself attempts to prove this postulate. He says that anyone who wants to prove it “must accept in advance such an axiom as Aristotle [*De Caelo* 1.5.271b 28 ff.] used in establishing the finiteness of the cosmos: If from a single point two straight lines making an angle are produced indefinitely, the interval between them when produced indefinitely will exceed any finite magnitude” (Proclus, p. 291). By means of this axiom, Proclus is then able to prove the Parallel Postulate, but assuming that the distance between two parallel lines is of a finite magnitude.

In a passage preserved in the commentary of Abu’l-ʿAbbās Fażl b. Ḥātem Nayrizi (fl. ca. 287/900) on the *Elements*, the Greek philosopher and commentator Simplicius (first half of the 6th century) quotes a proof of the Parallel Postulate made by his colleague Aḡānis, possibly the Athenian philosopher Agapius, who flourished about 511 (Lo Bello, pp. 224-29). Aḡānis first defines parallel lines as “those in one plane [such that] if they are extended with an endless extension, without limit, in both directions, together, the distance that is between them is always one distance” (Lo Bello, p. 158). This definition of parallel lines as equidistant straight lines will then enable him to prove Proposition I.29 of the *Elements*, as well as the Parallel Postulate.

The first Arabic mathematician who dealt with the Euclidean theory of parallels is ʿAbbās b. Saʿid Jawhari (fl. ca. 215/830), in a treatise, now lost, devoted to Euclid’s *Elements*. But his attempt to prove the Parallel Postulate has been preserved by Naṣir-al-Din Ṭusi (597-672/1201-74) in his “treatise that relieves from the doubt regarding parallel lines” (*al-Resāla al-šāfia ʿan al-šakk fi al-ḵoṭuṭ al-motawāzia*). Jawhari notably proves that parallel lines are equidistant; but he assumes implicitly that if a straight line falling on two straight lines makes the alternate angles equal to one another, then any other straight line which falls on the two straight lines will also make the alternate angles equal to one another. He then proves the Parallel Postulate (Jaouiche, pp. 24, 37-44, 137-44; Houzel, p. 170).

After Jawhari, Abu’l-Ḥasan Ṯābet b. Qorra (211-88/826-901) made two attempts to prove the Parallel Postulate. In his treatise on the proof of Euclid’s celebrated Postulate (*Maqāla fi borhān al-mosādara al-mašhura men Oqlides*), he admits as a principle that if two straight lines cut by another straight line diverge in one direction, they will converge in the other direction. This will enable him to prove that in case the alternate angles are equal, the two straight lines will be equidistant. He then proves the Parallel Postulate by means of the so-called “Axiom of Archimedes.” In the treatise on “The fact that two lines produced according to less than two right angles will meet” (*Fi anna al-ḵaṭṭayn eḏā oḵrejā elā aqall men zāwiatayn qāʾematayn eltaqayā*), Ṯābet b. Qorra introduces the concept of motion. He notably admits as a principle that any point on a solid that moves according to a uniform and rectilinear translation will describe a straight line. This enables him to produce two equidistant straight lines (Jaouiche, pp. 22-23, 45-56, 145-60; Houzel, p. 171).

Abu ʿAli Ḥasan b. Ḥasan b. Hayṯam (d. after 432/Sept. 1040) defines parallel straight lines through the concept of equidistance. In order to achieve this, he attempts to prove that if a finite straight line moves perpendicularly to a fixed straight line, then its extremity will describe a straight line parallel to the fixed line; this will enable him to prove the Parallel Postulate (Jaouiche, pp. 57-74, 161-84; Houzel, pp. 171-72).

Khayyam considers that the attempts of his predecessors to prove the Parallel Postulate were not satisfactory, in that each one of them had postulated something that was by no means easier to admit than the Postulate itself. He elaborates in particular on Ebn al-Hayṯam’s attempt, rejecting categorically the introduction of the concept of motion into geometry. Khayyam’s intention is to prove eight propositions, notably Proposition I.29 of the *Elements*, and the Parallel Postulate (Rashed and Vahabzadeh, 2000, pp. 185, 219-20, 225-27, 230-33). What makes his attempt particularly interesting is his philosophical position on this matter. Khayyam thinks that the error his predecessors made, while trying to prove the Parallel Postulate, is that they disregarded some of the principles taken from the philosopher (i.e., Aristotle). He believes that the Parallel Postulate should be proven taking as a starting point certain philosophical premises, which, in his opinion, are immediate consequences of the very notions of straight line and of rectilinear angle; for once these premises be taken as necessarily true, then the geometer can admit them without proof. These premises are: (1) Two straight lines that intersect will diverge while going away from the point of intersection (Proclus had already resorted to this premise, attributing it explicitly to Aristotle). (2) Two converging straight lines will intersect. (3) Two converging straight lines cannot diverge while going toward convergence, and vice-versa. Khayyam also assumes (while proving the third proposition) that parallel lines are equidistant; but since he does not give any explanation whatsoever, it is difficult to know whether he takes this as an obvious consequence of the second premise, or whether he considers, like some of his predecessors, that “parallel” and “equidistant” are synonymous (Rashed and Vahabzadeh, 2000, pp. 185, 224, 226).

It should be noted in this respect that, from Khayyam’s point of view, the fact that the second and third premises are mathematically equivalent to the postulate he intends to prove is not an issue at all. Khayyam is not really concerned with matters of mathematical equivalence; he is rather concerned with the fact that the second and third premises are immediate consequences of the concepts of straight line and rectilinear angle, whereas the Postulate is not; and this is why the Postulate should, in his opinion, be proven through them.

The gist of Khayyam’s argumentation is found in the proof of the third proposition. In it, he considers (Figure 1) a quadrilateral ABCD, in which the sides AC and BD are equal to each other and both perpendicular to the base AB. As a consequence of the first proposition he had just proven, the angles ACD and BDC will then be equal to each other.

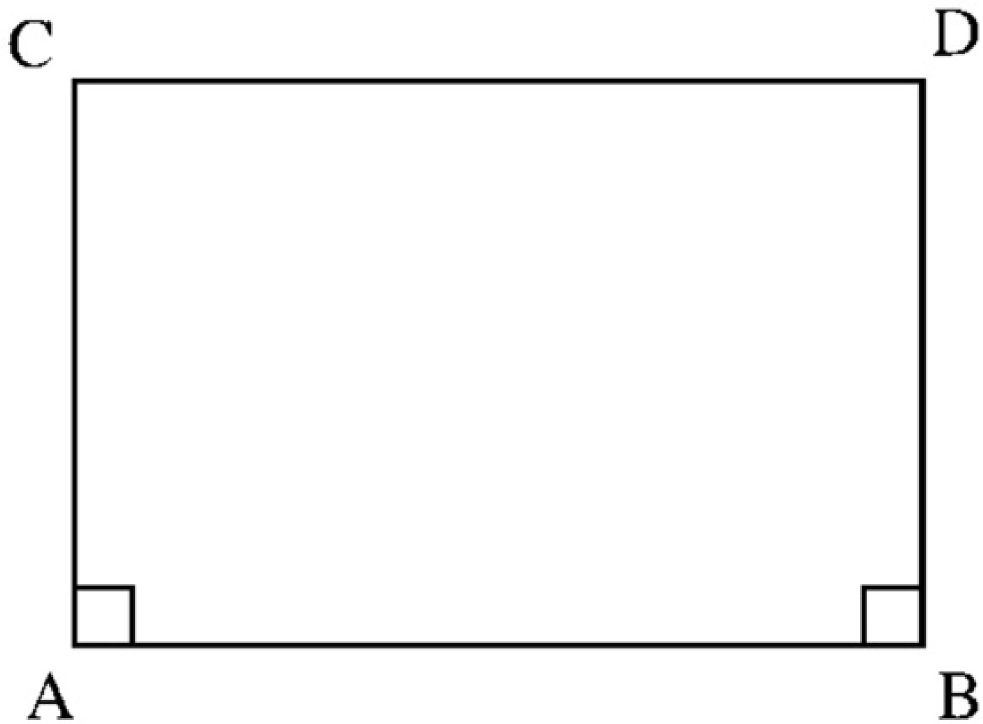


Figure 1. Diagram for Khayyam's proof of the third proposition.

He next examines successively the three possible cases, namely that the angles ACD, BDC are both right, both acute, or both obtuse. He first proves that, if one assumes that these angles are acute, then one will get two straight lines that cut another straight line at right angles and diverge on both sides of this straight line; and this contradicts the third premise. Likewise one will arrive at a contradiction assuming that these angles are obtuse. Therefore the angles ACD, BDC will necessarily be right angles. He can now easily prove Proposition I.29 of the *Elements*, as well as the Parallel Postulate (Rashed and Vahabzadeh, 2000, pp. 185-86, 226-30). Khayyam ends this part of his treatise by explaining that the eight propositions he has just proven should take the place of Proposition I.29 of the *Elements*, omitting, however, all the philosophical considerations he has elaborated upon, since these properly belong to the science of metaphysics, not to geometry (Rashed and Vahabzadeh, 2000, pp. 186, 233).

About two centuries later, Naṣir-al-Din Ṭusi took up in part Khayyam's ideas in two treatise: one is the treatise that relieves the doubt regarding parallel lines (*al-Resāla al-šāfia*), which is devoted to the proof of the Parallel Postulate and contains extensive passages from Khayyam's proof; the other is the Redaction of Euclid (*Taḥrir Oqlides*), a treatise devoted to Euclid's *Elements* in their entirety.

Traces of Khayyam's proof of the Parallel Postulate can still be found as late as the 18th century. In the first propositions of his *Euclides vindicatus* (Euclid freed of all blemish), the Jesuit mathematician Girolamo Saccheri (1667-1733) considers the very same quadrilateral ABCD, now known as the "quadrilateral of Saccheri," as well as the three possible cases regarding its equal angles ACD, BDC, which Saccheri calls, respectively, the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle.

*Concepts of ratio and proportionality*. Euclid had expounded in Book V of the *Elements* the theory of proportion applicable to every kind of magnitude (lines, surfaces, solids, time). The whole theory was based on the definitions found in the beginning of the Book, of which the following two were to play a prominent part: "3. A ratio is a sort of relation in respect of size between two magnitudes of the same kind. 5. Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order" (Heath, II, p. 114).

What should be noted regarding Definition V. 3 is that the meaning of the words "a sort of relation in respect of size" is nowhere explained in the *Elements*. This posed a problem as to how they should be interpreted, notably the Greek word *pēlikotēs*, which is variously translated as "size," "value," or "quantity."

Definition V.5 was the cornerstone of the theory, since it could apply to all magnitudes, whether commensurable or incommensurable, in contradistinction to the alternative definition of proportionality found in Book VII of the *Elements*, which, strictly speaking, applied only to numbers but could easily be extended to commensurable magnitudes, namely Definition VII.20: "Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth" (Heath, II, p. 278).

Definition V.5 posed some problems. First of all, the comparison of multiples of magnitudes did not seem to have any definite and obvious relationship with the concept of proportionality. Secondly, Euclid did not give any indication on how this definition had been conceived or established, so that nothing relevant could be found within the *Elements* themselves to enable mathematicians to unravel his intention. Finally, although it was meant to define the concept of "same ratio," Euclid's exposition did not permit to establish a relationship between Definition V.3 and Definition V.5 (Vahabzadeh, 2002, pp. 10-11).

No Greek commentary on Definition V.5 has come down to us (Euclid of Alexandria, tr. Vitrac, II, pp. 539-43); however, the situation is quite different with regard to Arabic mathematics. This Definition has indeed given rise to numerous commentaries in Arabic, whose aim was either to justify it by means of a proof, or to substitute for it another definition known as the "anthyphairetic definition of same ratio" (see, e.g., Plooij, pp. 48-56, 61-66). The latter definition consisted in applying to two homogeneous magnitudes a process usually known as the Euclidean algorithm, but which historians also call anthyphairesis after a Greek word meaning "alternating subtraction," "reciprocal subtraction." More precisely, given two homogeneous magnitudes, the lesser magnitude is subtracted from the greater a certain number of times, until one arrives at a remainder less than the lesser magnitude. Then this remainder is subtracted from the lesser magnitude a certain number of times, until one arrives at a second remainder less than the first remainder. Then one proceeds in the same manner with every pair of consecutive remainders. The sequence of natural numbers thus obtained can then be considered as a "characteristic" of the ratio of the two magnitudes. Now if the ratio of another pair of homogeneous magnitudes is characterized by the same sequence of numbers, then the four magnitudes are said to be in the same ratio, that is, they will be proportional (see, e.g., Plooij, pp. 57-60). For instance, let us assume that AB and CD (Figure 2) are two homogeneous magnitudes. Let us suppose that AB measures CD once, leaving ED less than AB; that ED measures AB three times, leaving FB less than ED; and that FB measures ED twice, leaving GD less than FB. If we proceed in the same manner with every pair of successive remainders, this process will then yield the sequence 1, 3, 2, …

Also consider KL and MN (Figure 3) to be another pair of homogeneous magnitudes. We suppose that KL measures MN once, leaving ON less than KL; that ON measures KL three times, leaving PL less than ON; and that PL measures ON twice, leaving QN less than PL, and proceed in the same manner with every pair of successive remainders. Now if the process applied to KL and MN yields the same sequence 1, 3, 2 …, then the ratio of AB to CD is said to be the same as the ratio of KL to MN.

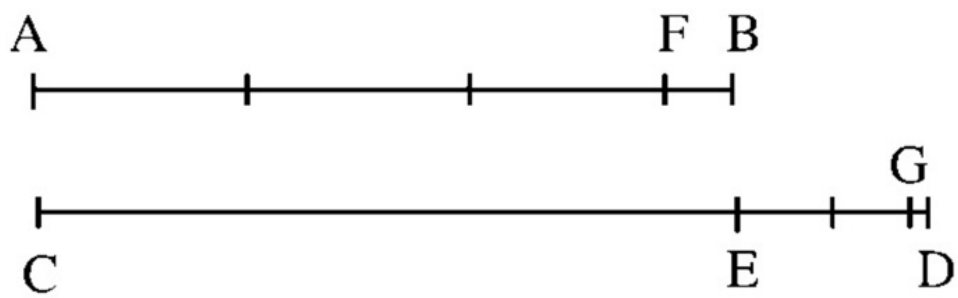


Figure 2. Illustration of anthyphairesis.

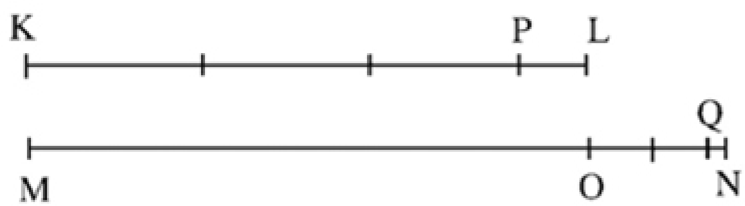


Figure 3. Illustration of the anthyphairetic definition of same ratio.

As noted by some historians (Plooij, p. 63; Gardies, pp. 80-81, 90-91; Vahabzadeh, 1997, pp. 253-57), one of the features of the anthyphairetic definition of same ratio is that, unlike Euclid's Definition V.5, it allows for each one of the ratios that make up a proportion to be considered independently of the other, which is necessary in order to give a meaning to the concept of ratio in itself, notably for regarding a ratio between any two magnitudes as a number (see below, *Compounding of ratios and irrational numbers*).

Euclid had already used the anthyphairetic process for finding the greatest common measure of two numbers and of two commensurable magnitudes (*Elements*, Propositions VII.2 and X.3). He also used it as a criterion for proving that two numbers are prime to one another, and that two magnitudes are incommensurable (Propositions VII.1 and X.2); he, however, did not use this process to define proportional magnitudes. In *Topics* VIII.3. 158b 29-35, Aristotle states: “It would seem that in mathematics also some things are not easily proved by lack of a definition, such as the proposition that the straight line parallel to the side which cuts the plane [i.e., the parallelogram] divides in the same way both the line and the area. But when the definition is stated, what was stated becomes immediately clear. For the areas and the lines have the same *alternating subtraction* (*antanairesis*); and this is the definition of the same proportion” (Thomas, I, p. 507). In his commentary on this passage, Alexander of Aphrodisias (ca. 210) adds: “For the definition of proportions which those of old times used is this: Magnitudes which have the same alternating subtraction (*anthyphairesis*) are proportional. But he [i.e., Aristotle] has called *anthyphairesis antanairesis*” (Thomas, I, p. 507). Many historians consider this as evidence of the existence in pre-Euclidean mathematics of a definition of proportional magnitudes based on anthyphairesis, and some have attempted to reconstruct such a pre-Euclidean theory of proportion; however, no trace of this anthyphairetic definition has been found in any extant Greek mathematical text (e.g., see Fowler, 1999a; Euclid of Alexandria, tr. Vitrac, II, pp. 515-23; and esp. Vitrac, 2002, pp. 158-74).

The first mathematical text in which the anthyphairetic definition of same ratio is explicitly mentioned is the Treatise On the Difficulty Concerning Ratio (*Resāla fi’l-moškel men amr al-nesba*) by Abu ʿAbd-Allāh Moḥammad b. ʿĪsā Māhāni (fl. 246/860). In it, Māhāni considers the ratio of two homogeneous magnitudes as the state occurring to each magnitude when measured by the other. Following a directive of Ṯābet b. Qorra, he characterizes this measure by means of the anthyphairetic process; two ratios will then be the same if this process yields the same sequence of numbers when applied to each pair of magnitudes (see above). Māhāni also states a definition of greater ratio based on anthyphairesis. He then proves that his definitions of same ratio and of greater ratio are equivalent to Definition V.5 and Definition V.7 (Euclid’s definition of greater ratio) respectively (Vahabzadeh, 2002, pp. 12-14, 31-40).

In his commentary on Euclid’s *Elements*, Abu’l-ʿAbbās Nayrizi (fl. ca. 287/900) has also interpreted Definition V.3 in terms of anthyphairesis; but, unlike his predecessor Māhāni, he considers that it is not necessary to prove Definition V.5 since, in his opinion, this definition belongs to the principles of Book V. Neither does he study the connection between the anthyphairetic definition of same ratio and Definition V.5 (Plooij, pp. 51-53, 61).

In the second part of his commentary, Khayyam intends to deal thoroughly with the concepts of ratio and proportionality between magnitudes, since in his opinion this matter had never been dealt with in a satisfactory and philosophical manner. Commenting upon Euclid's Definition V.3, Khayyam says that two things enter into the concept of ratio: the relation between the two magnitudes as to equality and inequality, and the size, or the magnitude, of this ratio. He explains that this concept was first found in natural numbers; that is, when one considers the numbers related to one another, one finds that they are either equal or unequal; and if they are unequal, then the smaller number will either be a part or parts of the larger one. For instance, since 3 measures 9 three times, 3 is one-third of 9, and the size of the ratio of 3 to 9 will be one-third; likewise 2 is two-sevenths of 7, and the size of ratio of 2 to 7 will be two-sevenths. When one considers this concept with regard to magnitudes, one will find, besides the preceding three possibilities, a fourth one, namely that the two magnitudes may be incommensurable, so that the less will neither be a part of the greater nor parts (Rashed and Vahabzadeh, 2000, pp. 188, 234-35).

He then recalls Euclid's Definition V.5, and adds: "But this does not manifest (*yonabbeʾ ʿan*) true proportionality. Do you not see that if a questioner inquires, saying: Four magnitudes are proportional according to Euclidean proportionality, and the first is the half of the second, will the third then be the half of the fourth, or not?" (Rashed and Vahabzadeh, 2000, p. 236, with correction). In other words, even though it is fairly easy to prove starting from Definition V.5 that the third magnitude will also be one-half the fourth (for according to Definition V.5 twice the third magnitude is equal to the fourth, so that the third is obviously one-half the fourth), yet this definition is not satisfactory in that it does not manifest an immediate property of proportional magnitudes, as in fact any true definition should. He calls the Euclidean concept of proportional magnitudes "common proportionality" and intends to speak of "true proportionality" (Rashed and Vahabzadeh, 2000, pp. 188, 236).

Khayyam's conception of ratio and proportionality is essentially the same as that of his predecessors Māhāni and Nayrizi. That is, given two magnitudes, they will either be equal, or the less will be a part or parts of the greater; and if the two magnitudes are incommensurable, their relation will then be characterized by means of anthyphairesis; whence it follows that two ratios will necessarily be the same if the anthyphairetical process applied to each pair of magnitudes yields the same sequence of numbers. Khayyam also defines the concept of greater ratio through anthyphairesis. He then proves that the anthyphairetic definitions of same ratio and of greater ratio are equivalent to Euclid's corresponding definitions. Consequently all properties of proportional magnitudes that had already been established within the framework of the Euclidean theory will remain valid within the framework of a theory based on the anthyphairetic definitions; therefore, these properties do not need to be proven again (Rashed and Vahabzadeh, 2000, pp. 188-89, 236-49).

*Compounding of ratios and irrational numbers.* We find in the beginning of Book VI of the *Elements* a definition (now considered as an interpolation) according to which a ratio is compounded of ratios when, the sizes of these ratios being multiplied together, they produce a certain ratio. But Euclid does not explain anywhere what he means by "the size of a ratio," not to mention the "multiplication" of these sizes. However, he does make use of the compounding of ratios in Propositions VI.23 and VIII.5 of the *Elements*. In each case, he admits without proof that, given three magnitudes or three natural numbers A, B, C, the ratio of A to C will be compounded of the ratio of A to B and of the ratio of B to C. It is this last statement, the one mathematicians use when compounding ratios, that has often been considered as a proposition that should be proven. This is in fact what Eutocius of Ascalon (b. ca. 480) applies himself to when commenting on Proposition II.4 of *On the Sphere and the Cylinder*, in which Archimedes (ca. 287-212 BCE) had used the preceding statement on the compounding of ratios. But although Eutocius intends to provide a general proof that this statement is valid for both natural numbers and magnitudes, he only considers ratios between numbers, so that his proof does not apply to ratios between incommensurable magnitudes (Heath, pp. 189-90, 247-48, 354; Youschkevitch, pp. 86-87; Netz, 2004a, pp. 312-15).

Bernard Vitrac has shown that Eutocius attempted later on to overcome the preceding limitation while commenting on Proposition I.11 of the *Conics* of Apollonius of Perga (b. ca. 262 BCE). Says Eutocius: "A ratio is said to be compounded of ratios when the sizes of the ratios multiplied by themselves produce something; it is being understood that 'size' is obviously said of the number to which the ratio is paronymous. On the one hand, it is with the multiples that the size can be a natural number [e.g., the size of the triple ratio is three]; with the other relations, the size will necessarily be a number and a part or parts [e.g., the size of the sesquialter ratio is one-and-a-half], unless, however, someone maintains that there also exist inexpressible relations, as are those between irrational magnitudes. And on the other hand, it is obvious that, in all the relations, this very size multiplied by the consequent term of the ratio will produce the antecedent" (Vitrac, 2000, Annexes, pp. 99-100). Eutocius then produces a proof essentially the same as the one in his commentary on Proposition II.4 of *On the Sphere and the Cylinder*, but this time he does not specify whether the ratios are numerical or not. He finally adds: "But the readers must not be worried by the fact that this has been demonstrated by arithmetical means, even if the Ancients indeed made use of these demonstrations by proportions that are more mathematical than arithmetical, and this because the object of the research is arithmetical; for ratios and sizes of ratios and multiplications of numbers belong first of all to numbers, and, from there, to magnitudes, in accordance with the saying: 'for these mathematical sciences appear to be sisters'" (Vitrac, 2000, p. 100; cf. Knorr, 1989, pp. 157-59).

Comments of Eutocius on the compounding of ratios provides the groundwork for illustrating the fundamental problem that arises in this context. Namely, that in Greek arithmetic a number is considered as a multitude of indivisible units (what we now call a natural number); consequently the size (*pēlikotēs*) of the ratio of two numbers or two commensurable magnitudes cannot, strictly speaking, be considered as a number, unless the antecedent of the ratio is a multiple of the consequent. Therefore, if one wants to consider as a number the ratio of any two numbers or commensurable magnitudes, it will then be necessary to consider a divisible unit, as in Greek logistic, which deals with concrete rather than theoretical units. In the first case, one will end up with what is called a whole or natural number, and in the second case with a fractional number. And in case one wants to consider as number the size of the ratio between two incommensurable magnitudes, one will end up with what is called an irrational number.

Now Khayyam sets out to prove the preceding statement on the compounding of ratios in the general case, that is, for any three magnitudes; and it is precisely in this context that he engages in a detailed study of the quantitative nature of ratio and brings forward the concept of irrational number.

For Khayyam, every ratio expresses a measure; that is, a certain magnitude is assumed as unit, and the other magnitudes of the same kind are related to it. For example, the meaning of "the ratio of three to five" is "three-fifths of a unit." In case there be given a ratio between two magnitudes A and B, he then considers the magnitude G such that its ratio to the unit is the same as the ratio of A to B. It is this magnitude G that will then express the measure (i.e., the size) of the ratio of A to B. Khayyam explains: "As to studying whether the ratio between magnitudes includes number in its essence, or whether it is inseparable from number, or whether it is joined to number from outside its essence because of something else, or whether it is joined to number because of something inseparable from its essence without requiring an extrinsic judgment: this is a philosophical study to which the geometrician must by no means devote himself," for this study is not incumbent upon the geometrician once he has "realized that a ratio between magnitudes is conjoined with something numerical or in the potentiality of number" (Youschkevitch, pp. 87-88; Rashed and Vahabzadeh, 2000, p. 251).

While proving the proposition in question, Khayyam goes back over these notions: "The magnitude G should not be regarded as being a line, or a surface, or a solid, or a time. On the contrary, it should be regarded as being abstracted in the intellect from these adjunct characters and as being attached to number: not as a true absolute number, for it may be that the ratio between A and B is not numerical, so that no two numbers can be found in accordance with their ratio" (Rashed and Vahabzadeh, 2000, p. 253). In this manner, he is able to reduce the compounding of ratios to the multiplication of the numbers that express their respective sizes. He also explains that the unit he considers is a divisible unit (in fact the unit considered by him, being a magnitude, is divisible ad infinitum); and it is only by assuming that a number such as 2 is composed of divisible units that one will be able to speak of "the irrational number √2," unlike the ancient Greeks, for whom a concept such as "the irrational number √2" did not seem to have had any meaning; they would only speak in this case of the ratio of two incommensurable lines, namely the ratio of the diagonal of a square to its side.

This is how Omar Khayyam, by discussing the connection between the concept of ratio and the concept of number, and by raising explicitly the theoretical problems related thereto, made a decisive contribution both to the theoretical study of the concept of irrational number, and to the understanding of its status as a mathematical entity in its own right. For although Khayyam's point of view (it is not up to the geometer to justify the connection of ratio and number once he has realized that such a connection exists) might seem mathematically defective, it corresponds in fact to the attitude ultimately adopted by mathematicians for many centuries. Such an attitude can be found, for example, in the beginning of Isaac Newton's *Universal Arithmetick*, where he asserts without any kind of justification: "*By* Number *we understand*, not so much a Multitude of Unities, as *the abstracted Ratio of anyQuantity, to another*

*Quantity of the same Kind, which we take for Unity*. And this *is threefold*; *integer*, *fracted*, and *surd*: *An* Integer, *is what is measured by Unity*; *a* Fraction, *that which a submultiple Part of Unity measures; and a* Surd, *to which Unity is incommensurable*" (Newton, p. 2; Youschkevitch, pp. 88-89).

(2) THE ESSAY ON THE DIVISION OF THE QUADRANT OF A CIRCLE

This essay has no title and is not dated; we only know that it was written prior to the treatise on algebra, since in the former, which deals only with one specific cubic equation, Khayyam alludes to the subject matter of the latter, namely, a full treatment of all cubic equations (for this essay, see Amir-Moez, 1961; Djebbar and Rashed).

The aim of this essay is to determine (Figure 4) on the quadrant AB of a given circle ABCD a point G, so that the radius AE is to the perpendicular GH as EH to HB. In order to achieve this, Khayyam uses the traditional method of analysis and synthesis: He first assumes that the problem has been solved, and then deduces certain properties that will enable him to construct the point G, which is looked for.

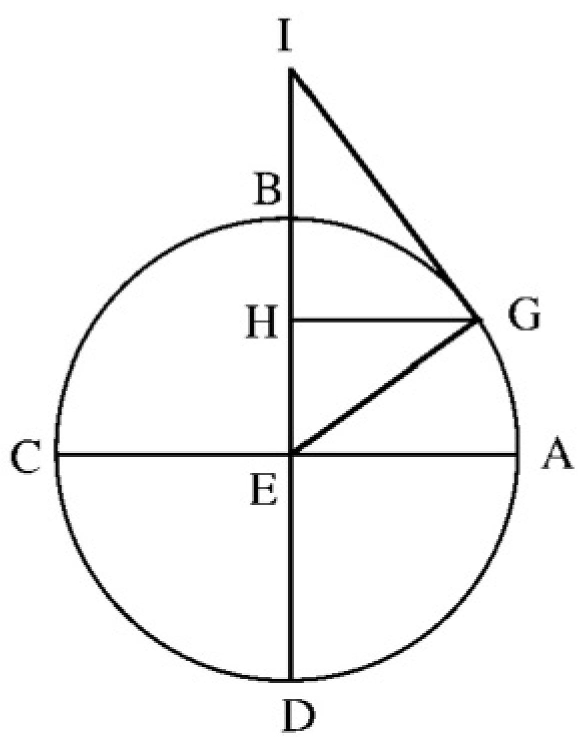


Figure 4. Diagram for Khayyam's untitled essay on the division of the

quadrant of a circle.

The first analysis leads to the determination of a rectangular hyperbola that passes through the center E of the circle; it is left unachieved because of its difficulty. In the second analysis, Khayyam assumes that the point G is known, and draws the tangent GI to the circle. He is thus led to the determination of the triangle EGI having a right angle at G.

After having examined certain properties of this triangle, he assumes that HG is a “thing,” that is, the unknown of an equation, also called “root” or “side,” and that EH is equal to 10. He is thus led to the resolution of the equation “a cube and two hundred things are equal to twenty squares and two thousand.” He then constructs the solution of this equation by means of a circle and a rectangular hyperbola. He is then able to construct the triangle EGI, and consequently the point G, which is looked for (Rashed and Vahabzadeh, 2000, pp. 97-107, 165-70, 174-79).

In the only known manuscript of this treatise (in the 1751 collection of Tehran University Library), Khayyam’s text is followed by a short problem that is attributed neither to Khayyam nor to anyone else. In it, the point G is determined at once as the intersection of the given circle and a rectangular hyperbola passing through point B, instead of point E as in Khayyam’s first analysis, and having as asymptotes CA produced and the perpendicular to CA drawn from point C.

Khayyam’s essay also contains an important digression on the basic concepts of algebra, and a classification of cubic equations. Khayyam first explains that what the algebraists call “squared-square,” “squared-cube,” “cubed-cube,” … (i.e., in modern notation, x4, x5, x6, …) cannot have any meaning in sensible things, so that these expressions should only be understood metaphorically. He then adds: “And as to things which are used by the algebraists, and which exist in sensible things and in continuous magnitudes, they are fourfold: number, thing, square and cube” (Rashed and Vahabzadeh, 2000, p. 171). He explains that number is something abstracted in the intellect from material things: It is a universal intelligible that cannot exist concretely unless associated with particular objects. As for the thing, its position in relation to magnitudes is that of the straight line. The square will of course be a square whose side is equal to the thing; and likewise the cube will be a cube whose side is equal to the thing (Rashed and Vahabzadeh, 2000, pp. 170-71).

Khayyam then gives a classification of cubic equations in which he follows the methodology inaugurated by Moḥammad b. Musā Ḵᵛārazmi. As is well known, Ḵᵛārazmi had written his treatise on algebra (*al-Jabr wa'l-moqābala*) during the caliphate of al-Maʾmun (r. 198-218/813-33). In it Ḵᵛārazmi had first introduced the basic notions used throughout his treatise, which he defined as the three kinds of numbers needed in algebraic calculations; these three kinds are "squares," "roots" (which he also calls "things"), and "simple number" (i.e., ax2, bx, and c respectively, where a, b, c are natural numbers or positive fractions). He then considered all the combinations between these three kinds, thus obtaining three equations between two terms (i.e., ax2 = bx, ax2 = c, bx = c), and three equations involving three terms (i.e., ax2 bx = c, ax2 c = bx, bx c = ax2). Ḵᵛārazmi explained how to solve each of these six equations once the number of the term of the highest degree had been reduced to one. He solved the equations between two terms through specific examples; but he gave the solution of those between three terms in the form of a general rule applicable to any equation of the same species, and justified each rule by means of a geometrical construction. He then applied these rules to the resolution of various sorts of problems, both theoretical and practical ( Ḵᵛārazmi, tr. Rosen, pp. 5-21, and passim).

Khayyam first recalls that the combinations between number, roots, and square yield six equations that the algebraists have already solved. He then considers all the combinations between number, roots, squares, and cube that yield third-degree equations. These equations are either simple or compound. Simple equations are those between two terms. Compound equations are those that involve more than two terms: they are either trinomial or quadrinomial. Khayyam is thus led to three simple equations (i.e., x3 = ax2, x3 = bx, x3 = c), nine trinomial equations (i.e., x3 ax2 = c, x3 ax2 = bx, x3 c = bx, x3 c = ax2, x3 bx = c, x3 bx = ax2, ax2 bx = x3, ax2 c = x3, bx c = x3), and seven quadrinomial equations (i.e., x3 = ax2 bx c, x3 bx c = ax2, x3 ax2 c = bx, x3 ax2 bx = c, x3 ax2 = bx c, x3 bx = ax2 c, x3 c = ax2 bx). Discarding those that can be reduced to an equation of a lesser degree, he ends up with fourteen cubic equations, all of which can only be worked out by means of conic sections (Rashed and Vahabzadeh, 2000, pp. 172-73).

He then informs us that nothing had come down to him from the ancients in relation to these fourteen cubic equations, and that Māhāni was the first who dealt with one of them. Māhāni was trying to solve the following lemma that Archimedes had used in Proposition II.4 of his treatise *On the Sphere and the*

*Cylinder*: Given (Figure 5) two lines DB and BZ, where DB is twice BZ, and given a point T on BZ, to cut DB at a point X so that XZ is to TZ as the square on DB to the square on DX (Rashed and Vahabzadeh, 2000, pp. 173).


Figure 5. Diagram for Māhāni's solution to Archimedes' lemma.

Although Archimedes had promised to show later on how to determine point X, the solution of this problem was never found in any of his writings. Eutocius, in his commentary on this proposition, reproduces in full a text he found "in a certain old book," and which could correspond to Archimedes' solution: In it the problem is solved geometrically by means of a parabola and a rectangular hyperbola (Netz, 2004a, pp. 318-30; idem, 2004b, pp. 16-29).

Māhāni thought of analyzing this lemma by means of algebra, and he was thus led to the equation "a cube and a number are equal to squares." He tried to solve it by means of conic sections, but was unable to find its solution; thus "he settled the matter by saying that it was impossible" (Rashed and Vahabzadeh, 2000, p. 173). Until Abu Jaʿfar Moḥammad Ḵāzen (d. between 350-60/961-71) finally solved it by means of conic sections. Then Abu Naṣr b. ʿErāq (10th-11th cent.) solved, also by means of conics, the equation "a cube and squares are equal to a number," to which he was led by analyzing algebraically a lemma that Archimedes had admitted in order to determine the side of the regular heptagon inscribed in a circle. Abu'l-Jud Moḥammad b. al-Layṯ (10th-11th cent.) solved a particular case of the equation "squares are equal to a cube and roots and a number," to which mathematicians were led by analyzing the following problem: To divide ten into two parts, so that the sum of their squares, added to the quotient of the division of the greater by the less, be seventy-two (Rashed and Vahabzadeh, 2000, pp. 173-74).

Thus, according to Khayyam's testimony, there are three cubics, to which he also adds the equation "a cube is equal to a number," that have already been solved "by our eminent predecessors" (Rashed and Vahabzadeh, 2000, p. 174). He ends his digression, adding that no one had discussed the remaining ten, nor given a classification of all cubics, and that he intends to compose a treatise that will include an exhaustive treatment of these (Rashed and Vahabzadeh, 2000, p. 174).

On the whole, it can be said that the main interest of this essay of Khayyam does not lie in Khayyam's resolution of the specific problem of the division of the quadrant of a circle, since this problem can be solved at once by choosing the appropriate hyperbola, but in that it provides us with an insight into Khayyam's methodology, and with important data relating to the history of cubic equations.

(3) THE TREATISE ON ALGEBRA

This is the *Maqāla fi'l-jabr wa'l-moqābala* (A treatise on algebra; lit. A treatise on restoration and comparison); one manuscript has instead the title *Resāla fi'l-barāhin ʿalā masāʾel al-jabr wa'l-moqābala* (A treatise on the demonstrations of the problems of algebra; Rashed and Vahabzadeh, 1999, p. 117; Woepcke, Ar. text, p. 1). In this undated treatise, Khayyam realizes the project already mentioned in his essay, that is, an exhaustive investigation of cubic equations. Apart from an introductory section, in which Khayyam takes up and expands the discussions already found in his essay, this treatise can be divided into three parts: the equations that can be solved by means of ruler and compass, that is, by means of Euclid's *Elements* and *Data*; the equations that can only be solved by means of conic sections, that is, by means of Apollonius's *Conics*; and the equations that involve the inverse of the unknown.

In the introduction to his treatise, Khayyam defines algebra as "a scientific art whose subject is absolute numbers and measurable magnitudes qua unknown but connected with something known which enables one to determine them" (Rashed and Vahabzadeh, 2000, pp. 112-13). In accordance with Aristotelian philosophy, what Khayyam here means by "absolute numbers" are natural numbers, that is, a discrete quantity; magnitudes are "a continuous quantity, of which there are four: the line, the surface, the solid, and time, as it is mentioned in a general way in the *Categories* [6, 4b20-25] and in detail in *First Philosophy* [*Metaphysics*, δ, 13, 1020a7-33]" (Rashed and Vahabzadeh, 2000, p. 113). Khayyam not only understands mathematical concepts in accordance with Aristotelian philosophy, he also insists on the fact that the proofs in his treatise are based essentially on the works of classical Greek geometers: "It must be realized that this treatise will not be understood except by someone who masters Euclid's work on the *Elements* and his work on the *Data*, as well as two Books of Apollonius's work on *Conics*; and that if someone is not well versed in any one of these three [works], he will in no way understand it" (Rashed and Vahabzadeh, 2000, p. 113; the same statement is reasserted on pp. 127, 142, 145).

As in his essay, Khayyam classifies the equations obtained by combining number, roots, squares, and cube. But he considers here all equations of the first, second, and third degree. He thus obtains six simple equations (i.e., c = x, c = x2, c = x3, bx = x2, bx = x3, ax2 = x3), twelve trinomial equations (i.e., x2 bx = c, x2 c = bx, bx c = x2, x3 ax2 = bx, x3 bx = ax2, x3 = bx ax2, x3 bx = c, x3 c = bx, c bx = x3, x3 ax2 = c, x3 c = ax2, c ax2 = x3), and seven quadrinomial equations (i.e., x3 ax2 bx = c, x3 ax2 c = bx, x3 bx c = ax2, x3 = bx ax2 c, x3 ax2 = bx c, x3 bx = ax2 c, x3 c = bx ax2); that is, a total of twenty-five equations.

The equations that can be solved by means of ruler and compass are the linear and quadratic equations, as well as the cubics that can be reduced to an equation of a lesser degree. The only linear equation is: "a number is equal to a root"; and its resolution is straightforward.

The resolution of quadratic equations is demonstrated both numerically and geometrically. The geometrical proof is achieved through the introduction of a unit length; this allows Khayyam to represent the terms of any quadratic equation by rectangular figures, so that the original algebraic equation translates into an equation between rectangles and squares, that is, between geometrical magnitudes; in that manner, Khayyam is able to apply the results established in Euclid's *Elements* and *Data* (Rashed, 1997, p. 44).

For instance, the simple equation "a number is equal to a square" is solved in the following manner: the numerical solution is found by extracting the square root of the number. To solve the equation geometrically, Khayyam first assumes (Figure 6) that the straight line AC is equal to the unit, and draws AB equal to the given number and perpendicular to AC; the measure of the rectangle AD will then be the given number. It is thus required to construct a square E equal to the given rectangle AD, and this construction is shown in Proposition II.14 of the *Elements*. The side of the square E will then be given, as shown in Proposition 55 of the *Data*, and will be the geometrical solution of the equation.

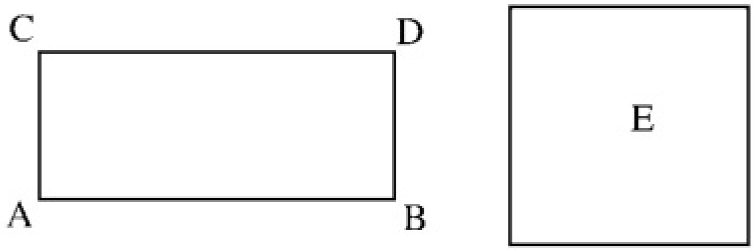


Figure 6. Diagram for Khayyam's treatise on algebra: finding a square equal to a rectangle.

Trinomial equations of the second degree are solved numerically as in Ḵᵛārazmi's treatise, that is, by means of a general rule applicable to all equations of the same species. Khayyam solves these equations geometrically in the same manner as his predecessors, that is, analytically; but he also adds a synthetic proof.

Let us consider, for instance, the equation "a square and ten roots are equal to thirty-nine." In order to find the numerical solution, he states the following rule: "Multiply half the number of the roots into itself, add the product to the number, and subtract from the root of the sum half the number of the roots. The remainder will then be the root of the square" (Rashed and Vahabzadeh, 2000, p. 120; cf. Ḵˇārazmi, tr., Rosen, p. 8). The fact that "number" here means "natural number" is clearly implied by the statement that follows: "Numerically, these two conditions are necessary: the first one of them, that the number of the roots be an even number, so that it may have a moiety; and the second, that the sum of the square of half the number of the roots and the number be a square number. For otherwise the problem would then be impossible numerically" (Rashed and Vahabzadeh, 2000, p. 120). In other words, here Khayyam discards both fractional and irrational numbers.

As for the geometrical solution, Khayyam provides three different proofs. The first proof is based on Proposition II.6 of the *Elements* and is virtually the same as the one produced by Ṯābet b. Qorra in his authentication of algebraic problems by means of geometrical proofs (Ṯābet's proof in Rashed, 2009, pp. 160-65); the second proof reproduces that of Ḵᵛārazmi's. Both proofs are analytical in that it is assumed in both of them that the side of the square being looked for is given, implying that this square has already been constructed. Khayyam produces a third synthetic proof. He supposes (Figure 7) that the line AB is equal to 10, and that the rectangle E is equal to 39. He then applies to AB a rectangle BD equal to E and exceeding AB by a square AD, as shown in Proposition VI.29 of the *Elements*. The side AC of the square will be given, as shown in Proposition 59 of the *Data*. Thus the line AC will be the root looked for.

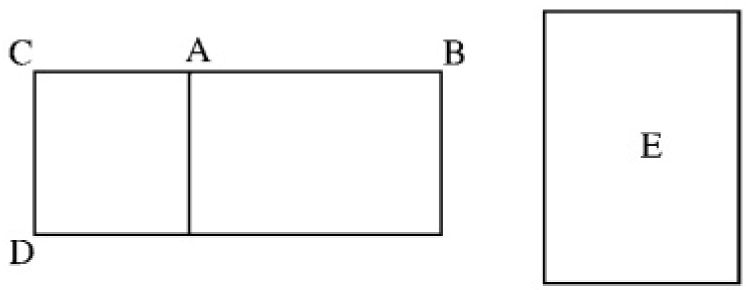


Figure 7. Diagram for Khayyam's third solution of equation "a square and ten roots are equal to thirty-nine."

The equations that can only be solved by means of conics are the fourteen cubics that cannot be reduced to an equation of a lesser degree (see above). Khayyam informs us that neither he nor his predecessors were able to solve them numerically, adding that "possibly someone else will come to know it after us" (Rashed and Vahabzadeh, 2000, p. 114; the numerical rules for solving cubic equations were discovered in the 16th century by the Italian algebraists Scipione del Ferro and Niccolo Fontana Tartaglia). Khayyam solves these equations only geometrically, using the properties of conic sections. The construction of the solutions of these fourteen cubics constitutes the bulk of Khayyam's treatise.

As was the case with quadratic equations, the construction of the solutions is achieved through the introduction of a unit length; this allowed Khayyam to represent each of the terms of a cubic equation by a rectangular parallelepiped, so that the original algebraic equation would translate into an equation between solids, that is, between geometrical magnitudes. This way, Khayyam was able to base his demonstrations on Euclid's *Elements* and *Data*, and on Apollonius's *Conics*.

Khayyam first proves three lemmas. The first lemma enables him to construct a cube equal to a given rectangular parallelepiped; the second and third lemmas are used each time Khayyam needs to represent the number in a cubic equation as a solid having either a given base or a given height. He is now able to solve geometrically the equation “a cube is equal to a number” (its numerical solution being the cube root of the number). He constructs (Figure 8) a rectangular parallelepiped ABCD whose base AC is the square of the unit and whose height BD is equal to the given number. It is thus required to construct a cube KHIL equal to ABCD.

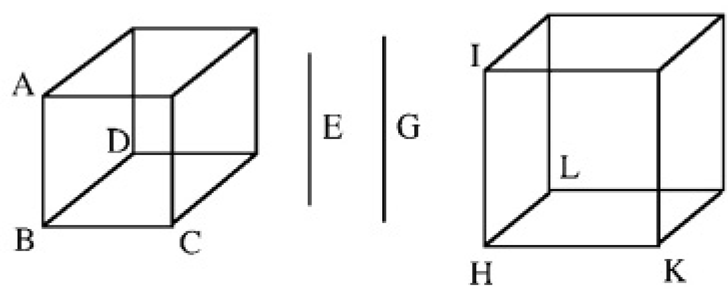


Figure 8. Diagram for Khayyam’s construction of a cube equal to a parallelepiped.

He takes two lines (E and G) that are mean proportionals between AB, BD (i.e., between the unit and the given number) as shown in the first lemma, and proves that the cube KHIL whose side HI is equal to E will then be equal to ABCD, that is, to the given number. Therefore, the side HI will be the solution of the equation.

Each one of the remaining cubic equations is solved by means of two single-branch conics. Khayyam investigates in each case the number of points at which these conics intersect or touch (not considering their vertices): the equation will accordingly have either one or two solutions. In some cases, however, the conics neither intersect nor touch, and the equation is “impossible”: it cannot be solved geometrically (Rashed and Vahabzadeh, 2000, pp. 130-56).

In this part of his treatise, Khayyam only produces synthetic proofs, in which he displays a full mastery of classical Greek geometry. However, the solution of each equation was probably found by means of an analysis; Roshdi Rashed has reconstructed such an analysis for the equation “a cube and sides are equal to a number" (i.e., x3 bx = c) by reverting the order of Khayyam's synthetic proof (Rashed, and Vahabzadeh, 2000, p. 37). It amounts to the following, when expressed in the mathematical language used by Khayyam: let AB be the side of a square MB equal to the number of the sides (i.e., to b). We construct (Figure 9) a rectangular parallelepiped whose base is MB and which is equal to the given number (i.e., to c), as shown in the second lemma; and let its height be BC.

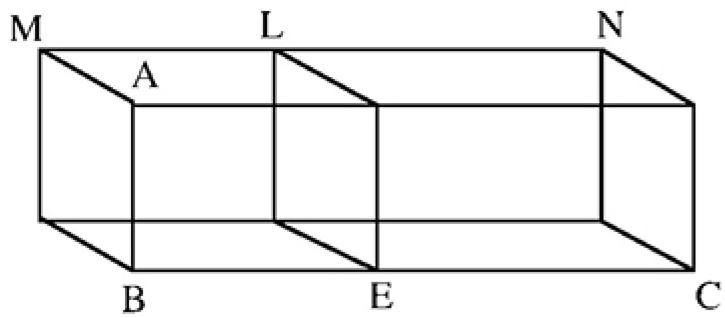


Figure 9. Diagram 1 for Khayyam's analysis of the equation "a cube and sides are equal to a number."

Now we assume that the problem has been solved, and that BE is the side of the cube being looked for (i.e., x); and we complete the square EL. The parallelepiped EM will be equal to the sides (i.e., to bx), but the parallelepiped BN is equal to the number (i.e., to c). Therefore, the remaining parallelepiped EN will be equal to the cube of BE (i.e., c – bx = x3), since BE is by hypothesis the root of the equation. In other words, the solid whose base is the square of AB and whose height is EC is equal to the cube whose side is BE. Therefore their bases will be reciprocally proportional to their heights, and the square of AB is to the square of BE as BE is to CE.

Now (Figure 10) let ED be a line perpendicular to BC and such that AB is to BE as BE to ED. Therefore, AB will be to ED in the duplicate ratio of AB to BE, that is, as the square of AB to the square of BE. But the square of AB is to the square of BE as BE to EC. Therefore, AB is to ED as BE to EC; and alternately AB is to BE as ED to EC. But AB is to BE as BE to ED. Therefore, BE is to ED as ED to EC; therefore the square of ED is equal to the product of BE and EC. Consequently the point D is on a circle whose diameter is BC. Also since AB is to BE as BE to ED, the square of BE will be equal to the product of AB and ED. Therefore, the square of DG is equal to the product of AB and BG. Consequently, the point D is also on a parabola whose vertex is B, whose axis is BG, and whose erect side is AB.

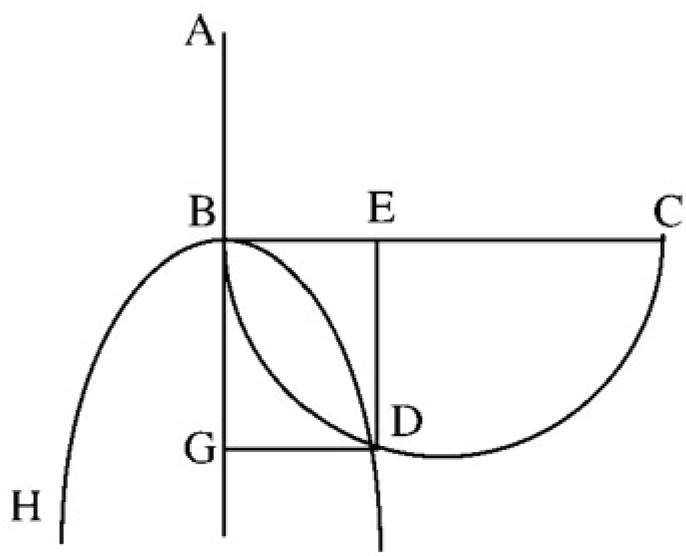


Figure 10. Diagram 2 for Khayyam's analysis of the equation "a cube and sides are equal to a number."

Thus the analysis has led to the determination of the intersection of a semicircle and a parabola. In the synthesis, Khayyam constructs the parabola HBD and the semicircle BDC, draws from the point of intersection D the line DE perpendicular to BC, and then proves that BE is the side of the cube looked for. According to Rashed, the remaining cubics were solved using the same method (Rashed and Vahabzadeh, 2000, pp. 37, 130-32); therefore, the previous analysis probably gives us an insight into the very process that led Khayyam to the resolution of third-degree equations.

The last part of Khayyam's treatise is devoted to equations involving the inverse of the unknown (i.e., x-1). In order to solve an equation like that, Khayyam takes the inverse as a new unknown and is thus led to one of the twenty-five equations previously studied. He then finds the solution of the latter equation and, taking its inverse, obtains the solution of the original equation. In this part of his treatise, Khayyam departs from the Euclidean rigorous methodology exhibited in the resolution of quadratic and cubic equations; for he only solves here particular equations and does not produce any proofs. Besides, he does not restrict the number concept to natural numbers as before, and mentions explicitly fractional numbers (Rashed and Vahabzadeh, 2000, pp. 156-59). Thus the fact that Khayyam had discarded fractions in the preceding parts of his treatise appears to be deliberate, and must have been due to his desire to follow, when possible, the Euclidean conception of number, that is, a multitude composed of indivisible units. But this would have been virtually impossible when dealing with equations involving the inverse of the unknown.

We have already noted that one of the most striking features of Khayyam's treatise on algebra is the geometrical nature of its argumentation. Of course, the twenty-five equations he intends to solve are in themselves algebraic concepts, and his classification could have hardly been conceived without the previous work of Ḵᵛārazmi; but once translated into a relation between geometrical figures, these equations are dealt with in a purely Euclidean manner. Also Khayyam constantly speaks of the product of a rectangle and a straight line, where Euclid would speak of the parallelepipedal solid with the rectangle as base and the straight line as height (e.g., Rashed and Vahabzadeh, 2000, pp. 119, 125-26; Heath, III, pp. 345-47); this terminology, however, is not altogether foreign to Greek geometry, for it is also found in Eutocius's commentary on Proposition II.4 of *On the Sphere and the Cylinder*, and even in Archimedes' alternate proof to Proposition II.8 of the same (Netz, 2004a, pp. 227-31, 320 ff.; idem, 2004b, pp. 97-120; see also 2004b, pp. 164-65). On the whole, it appears that in this treatise Khayyam made a deliberate return to the rigorous methods of Greek geometers, and that in this way he was able to build up the theory of quadratic and cubic equations on solid foundations.

(4) THE TREATISE ON THE EXTRACTION OF THE NTH ROOT OF NUMBERS

Apart from the preceding works, Khayyam also wrote an arithmetical work to which he alludes in his Treatise on algebra: "The Indians have methods for determining the sides of squares and cubes based on a restricted induction, that is, on the knowledge of the squares of the nine figures—I mean the square of the unit, of two, of three, and so on—and likewise of their product one into the other—I mean the product of two into three, and so on. And we have written a book to demonstrate the correctness of these methods and the fact that they fulfill the requirements; and we have increased the kinds thereof, I mean the determination of the sides of the squared-square, of the squared-cube, of the cubed-cube, whatever degree it may reach. And no one did it before us. But these demonstrations are only numerical demonstrations based on the arithmetical Books of the *Elements*" (Rashed and Vahabzadeh, 2000, pp. 116-17, with correction). This book has not come down to us and is known only through the preceding quotation. However, MS Or. 199 in the Leiden University Library (which also contains a copy of the Commentary on Euclid's *Elements*) lists on its title page, without including it, a work by Khayyam entitled *Moškelāt al-ḥesāb* (The difficulties of arithmetic). This work may correspond to his treatise on the extraction of nth roots (Rosenfeld and Youschkevitch, 1973, pp. 325-26; Youschkevitch, pp. 76, 80).

# BIBLIOGRAPHY


- Editions and translations.
- *Resāla fi šarḥ mā aškala men moṣādarāt Ketāb Oqlides*: ed., T. Erani (Arāni), Tehran, 1936; ed. and tr, Jalāl-al-Din Homāʾi, in idem, *Ḵayyāmi-nāma* I, Tehran, 1967 (pp. 177-222, 225-80); tr. Ali R. Amir-Moez, as "Discussion of Difficulties in Euclid by Omar Ibn Abrahim al-Khayyami (Omar Khayyam)," *Scripta Mathematica* 24/4, 1959, pp. 275-303; repr. in Fuat Sezgin, ed., *ʿUmar al-Khayyām, Abu'l-Fatḥ ʿUmar Ibn Ibrāhīm (d. c. 526/1131): Texts and Studies*, Islamic Mathematics and Astronomy 45-46, 2 vols., Frankfurt am Main, 1998, II, pp. 293-321 (an incomplete and inaccurate translation); ed. A. I. Sabra, in idem, "Omar Khayyām: Explanation of the Difficulties in Euclid's Postulates," Ph.D. diss., Alexandria, 1961; tr. Ahmed Djebbar, as "L'Emergence du concept de nombre réel positif dans l'épître d'al-Khayyām (1048-1131) sur l'explication des prémisses problématiques du livre d'Euclide (Introduction et traduction française)" *Prépublication* 97, no. 39, Université de Paris-Sud, 1997; idem, tr., as "Epître d'Omar Khayyām sur l'explication des prémisses problématiques du livre d'Euclide (traduction française)," *Farhang* 14, 2002, pp. 79-136 (revised French version of the preceding tr.).
- *Maqāla fi'l-jabr wa'l-moqābala*: tr. Daoud S. Kasir, as *The Algebra of Omar Khayyam*, New York, 1931, repr. in Fuat Sezgin, ed., *ʿUmar al-Khayyām, Abu'l-Fatḥ ʿUmar Ibn Ibrāhīm (d. c. 526/1131): Texts and Studies*, Islamic Mathematics and Astronomy 45-46, 2 vols., Frankfurt am Main, 1998, I, pp. 261-392; tr. H. J. J. Winter and W. ʿArafat, as "The Algebra of ʿUmar Khayyām," *JRASB, Science* 16/1, 1950, pp. 27-78, repr. in Fuat Sezgin, ed.,

*ʿUmar al-Khayyām, Abu'l-Fatḥ ʿUmar Ibn Ibrāhīm (d. c. 526/1131): Texts and Studies*, Islamic Mathematics and Astronomy 45-46, 2 vols., Frankfurt am Main, 1998, II, pp. 241-92; ed. and tr. Ahmed Djebbar and Roshdi Rashed, as *L'Œuvre algébrique d'al-Khayyām*, Alepo, 1981 (also an ed. and tr. of the essay on the division of the quadrant of a circle); ed. and tr., Franz Woepcke, *L'Algèbre d'Omar Alkhayyâmî*, Paris, 1851; repr. in Fuat Sezgin, ed., *ʿUmar al-Khayyām, Abu'l-Fatḥ ʿUmar Ibn Ibrāhīm (d. c. 526/1131): Texts and Studies*, Islamic Mathematics and Astronomy 45-46, 2 vols., Frankfurt am Main, 1998, I, pp. 1-206 (the editio princeps); tr. Roshdi Khalil, as *An Essay by the Unikely Wise ʿAbel Fath Omar Bin Al-Khayyam on Algebra and Equation*: *Algebra wa al-Muqabala*, Reading, UK, 2008. Ali R. Amir-Moez, tr., "A Paper of Omar Khayyam," *Scripta Mathematica* 26/4, 1961, pp. 323-37; repr. in Fuat Sezgin, ed., *ʿUmar al-Khayyām, Abu'l-Fatḥ ʿUmar Ibn Ibrāhīm (d. c. 526/1131): Texts and Studies*, Islamic Mathematics and Astronomy 45-46, 2 vols., Frankfurt am Main, 1998, II, pp. 323-37 (tr. of Khayyam's essay on the division of the quadrant of a circle).

- Roshdi Rashed and Bijan Vahabzadeh, ed. and tr., *al-Khayyām mathématicien*, Paris, 1999 (ed. and tr. of *Maqāla fi'l-jabr wa'l-moqābala,* of the essay on the division of the quadrant of a circle, and of *Resāla fi šarḥ mā aškala …*).
- Idem, *Omar Khayyam The Mathematician*, New York, 2000 (an Eng. version of *al-Khayyām mathématicien*; the mathematical works have been translated directly from the Arabic, and the introductions and commentaries from the French version).
- Idem, *Riāżiyāt ʿOmar al-Ḵayyām*, Beirut, 2005 (an Ar. version of *al-Khayyām mathématicien*). Boris A. Rosenfeld, tr., *Omar Khayyam, Traktaty*, ed. V. S. Segal and Adolf P. Youschkevitch, Moscow, 1961 (containing a facsimile of the manuscripts of Khayyam's mathematical and philosophical treatises, with Russ. tr.).
- Other primary sources.
- Apollonius of Perga, *Kōnika*, ed. and tr. Roshdi Rashid, Micheline Decorps-Foulquier, and Michel Federspiel, as *Coniques*, Berlin and New York, 2010 (Greek and Arabic texts).
- Rasmus O. Besthorn et al., eds., *Codex Leidensis 399, 1: Euclidis Elementa ex Interpretatione al-Hadschdschadschii cum Commentariis Al-Narizii*, Hauniae (Copenhagen), 1893-1932 (edition of the Arabic text with Latin translation and notes); repr. in Fuat Sezgin, ed., *Islamic Mathematics and Astronomy* XIV-XV, Frankfurt am Main, 1997.

- Gregg De Young, “Al-Jawharī’s Additions to Book V of Euclid’s Elements,” *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* 11, 1997, pp. 153-78.
- Euclid of Alexandria, *Stoicheia,* tr. and comm. Bernard Vitrac, with general introd. by Maurice Caveing, as *Les Éléments*, 4 vols., Paris, 1990-2001.
- Thomas L. Heath, tr., *The Thirteen Books of Euclid’s Elements*, tr. from the text of Heiberg, with introd. and comm., 2nd ed., 3 vols., Cambridge, 1926; repr., New York, 1956.
- Khalil Jaouiche, *La Théorie des parallèles en pays d’Islam: contribution à la préhistoire des géométries non-euclidiennes*, Paris, 1986.
- Moḥammad b. Musā Ḵᵛārazmi, *al-Moḵtaṣar fi ḥesāb al-jabr wa’l-moqābala*, ed. and tr. Franz A. Rosen, as *The Algebra of Mohammed ben Musa*, London, 1831, repr. New York, 1986; ed. and tr. Roshdi Rashed, as *Al-Khwārazmi*: *Le commencement de l’algèbre*, Paris, 2007; idem, ed. and tr., as *Al-Khwārazmi*: *The Beginnings of Algebra*, London and Beirut, 2009.
- Abu’l-ʿAbbās Fażl b. Ḥātem Nayrizi, *Šarḥ ketāb Oqlides*, tr. Anthony Lo Bello, as *The Commentary of Al-Nayrizi on Book I of Euclid’s Elements of Geometry, with an Introduction on the Transmission of Euclid’s Elements in the Middle Ages*, Boston and Leiden, 2003.
- Reviel Netz, *The Works of Archimedes*, tr. together with Eutocius’ commentaries, with commentary and critical edition of the diagrams, I: *The Two Books On the Sphere and the Cylinder*, Cambridge, 2004a.
- Isaac Newton, *Universal Arithmetick, Or, A Treatise of Arithmetical Composition and Resolution*, tr. Joseph Ralphson, and Revised and Corrected by Samuel Cunn, London, 1769.
- Proclus, *A Commentary on the First Book of Euclid’s Elements*, tr. with Introd. and notes by Glenn R. Morrow, Princeton, 1970.
- Roshdi Rashed, ed., *Thābit ibn Qurra: Science and Philosophy in Ninth-Century Baghdad*, Berlin and New York, 2009.
- Ivor Thomas, *Greek Mathematics*, 2 vols., Cambridge, Mass., and London, 1939-41; repr. with additions and revisions, 1980-93.
- Bijan Vahabzadeh, “Al-Māhānī’s Commentary on the Concept of Ratio,” *Arabic Sciences and Philosophy* 12, 2002, pp. 9-52.
- Secondary sources and studies.
- Jaʿfar Āqāyāni Čāwoši, ed., *Farhang*: *wiža-ye bozorgdāšt-e Ḵayyām*, 3 vols., Tehran, 2000-2005 (a collection of recent papers on all aspects of Khayyam’s work).
- Yvonne Dold-Samplonius, “Al-Māhānī,” in Helaine Selin, ed.,

*Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*, 2nd ed., Berlin, Heidelberg, and New York, 2008, pp. 141-42.

- Charles-Henri de Fouchécour and Boris A. Rosenfeld, "ʿUmar Khayyām," in *EI*2, X, 2000, pp. 827-34.
- David H. Fowler, *The Mathematics of Plato's Academy*: *A New Reconstruction*, 2nd ed., Oxford, 1999a. Idem, "Inventive Interpretations," *Revue d'histoire des mathématiques* 5, 1999b, pp. 149-53.
- Jean-Louis Gardies, *L'Héritage épistémologique d'Eudoxe de Cnide: un essai de reconstitution*, Paris, 1988.
- Jan P. Hogendijk, "Anthyphairetic Ratio Theory in Medieval Islamic Mathematics," in Yvonne Dold-Samplonius, Joseph W. Dauben, Menso Folkerts, and Benno Van Dalen, eds., *From China to Paris: 2000 Years Transmission of Mathematical Ideas*, Stuttgart, 2002, pp. 187-202.
- Christian Houzel, "Histoire de la théorie des parallèles," in Roshdi Rashed, ed., *Mathématiques et philosophie de l'antiquité à l'âge classique: hommage à Jules Vuillemin*, Paris, 1991, pp. 163-79.
- Wilbur R. Knorr, *The Evolution of the Euclidean Elements*: *A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*, Dordrecht and Boston, 1975.
- Idem, *Textual Studies in Ancient and Medieval Geometry*, Boston, Basel, and Berlin, 1989.
- Richard Lorch, "The Arabic Transmission of Archimedes, *Sphere and Cylinder* and Eutocius' Commentary," *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* 5, 1989, pp. 94-114.
- Ḡolām-Ḥosayn Moṣāḥab, *Ḥakim ʿOmar Ḵayyām be-ʿonwān-e ʿālem-e jabr*, Tehran, 1960.
- John E. Murdoch, "The Medieval Language of Proportions: Elements of the Interaction with Greek Foundations and the Development of New Mathematical Techniques," in Alistair C. Crombie, ed., *Scientific Change*, London, 1963, pp. 237-71.
- Reviel Netz, *The Transformation of Mathematics in the Early Mediterranean World: From Problems to Equations*, Cambridge and New York, 2004b.
- Jeffrey A. Oaks, "Al-Khayyām's Scientific Revision of Algebra," *Suhayl* 10, 2011, pp. 47-75.
- Edward B. Plooij, *Euclid's Conception of Ratio and His Definition of Proportional Magnitudes as Criticized by Arabian Commentators*, Rotterdam, 1950; repr. in Fuat Sezgin, ed., *Islamic Mathematics and*

*Astronomy* XIX, Frankfurt am Main, 1997, pp. 167-243.

- Roshdi Rashed, *Entre arithmétique et algèbre: recherches sur l'histoire des mathématiques arabes*, Paris, 1984; tr. Roshdi Rashed, as, *The Development of Arabic Mathematics: Between Arithmetic and Algebra*, Dordrecht, Boston, and London, 1994.
- Idem, "L'algèbre," in Roshdi Rashed and Régis Morelon, eds., *Histoire des sciences arabes* II, Paris, 1997, pp. 31-54.
- Boris A. Rosenfeld and Adolf P. Youschkevitch, "Al-Khayyāmī," in Charles C. Gillispie, ed., *Dictionnary of Scientific Biography*, 16 vols., New York, 1970-80, VII, 1973, pp. 323-34.
- Idem, revised and enlarged by Hélène Bellosta, "Géométrie," in Roshdi Rashed and Régis Morelon, eds., *Histoire des sciences arabes* II, Paris, 1997, pp. 122-62.
- Fuat Sezgin, *Geschichte des arabischen Schrifttums* V, Leiden, 1974.
- Idem, ed., *ʿUmar al-Khayyām, Abu'l-Fatḥ ʿUmar Ibn Ibrāhīm (d. c. 526/1131): Texts and Studies*, Islamic Mathematics and Astronomy 45-46, 2 vols., Frankfurt am Main, 1998 (a collection of classical papers on the mathematical and astronomical works of Khayyam).
- David Eugene Smith, "Euclid, Omar Khayyâm, and Saccheri," *Scripta Mathematica* 3, 1935, pp. 5-10; repr. in Fuat Sezgin, ed., *Islamic Mathematics and Philosophy* XIX, Frankfurt am Main, 1997, pp. 1-6.
- Bijan Vahabzadeh, "Al-Khayyām's Conception of Ratio and Proportionality," *Arabic Sciences and Philosophy* 7, 1997, pp. 247-63.
- Idem, "ʿUmar al-Khayyām and the Concept of Irrational Number," in Régis Morelon and Ahmad Hasnawi, eds., *De Zénon d'Elée à Poincaré: Recueil d'études en hommage à Roshdi Rashed*, Louvain and Paris, 2004, pp. 55-63.
- Bernard Vitrac, "ʿOmar Khayyām et Eutocius: les antécédents grecs du troisième chapitre du commentaire sur certaines prémisses problématiques du livre d'Euclide," *Farhang* 12, Tehran, 2000, pp. 51-105.
- Idem, "ʿOmar Khayyām et l'anthyphérèse: étude du deuxième livre de son commentaire sur certaines prémisses problématiques du livre d'Euclide," *Farhang* 14, Tehran, 2002, pp. 137-92. (Vitrac's papers are available online at https://hal.archives-ouvertes.fr/hal-00174927 and https://hal.archives-ouvertes.fr/hal-00174930 respectively.)
- Adolf P. Youschkevitch, *Les Mathématiques arabes (VIIIe–XVe siècles)*, Paris, 1976.